



Exponential stability in mean square of impulsive stochastic difference equations with continuous time[☆]

Jianhai Bao^{a,b,*}, Zhenting Hou^a, Fuxing Wang^c

^a School of Mathematics, Central South University, Changsha, Hunan 410075, PR China

^b Guangxi Institute of Technology, Liuzhou 545006, PR China

^c Jining Vocational and Technical College, Jining, Shandong 272037, PR China

ARTICLE INFO

Article history:

Received 15 August 2007

Received in revised form 24 August 2008

Accepted 28 August 2008

Keywords:

Exponential stability in mean square

Impulsive

Stochastic difference equation

Continuous time

Difference inequality

ABSTRACT

So far there have been few results presented on the exponential stability in mean square for impulsive stochastic difference equations with continuous time. The main aim of this work is to close this gap. Unlike earlier studies, ours does not make use of general methods such as Lyapunov methods, Itô formula methods and so forth. However, we obtain the desired result by establishing a difference inequality with continuous time. Moreover, the result obtained can be applied to stochastic difference equations, without impulsive effects, with continuous time. Finally, we construct an example to illustrate the effectiveness of our result.

© 2008 Elsevier Ltd. All rights reserved.

1. Introduction

Difference equations with continuous time are difference equations in which the unknown function is a function of continuous time. (The term “difference equation” is usually used for difference equations with discrete time.) In practice, time is often involved as the independent variable in difference equations with continuous time. In view of this fact, we may refer to them as *difference equations with continuous time*. Difference equations with continuous time appear as natural descriptions of observed evolution phenomena in many branches of natural sciences (see, e.g., [9,10] and references therein). Deterministic and stochastic difference equations with continuous time are very popular with researchers (see, e.g., [11–14] and references therein).

Impulsive effects exist in many evolution processes in which states are changed abruptly at certain moments of time, involved in such fields as medicine and biology, economics, mechanics, electronics (see [1] and reference therein). However, in addition to impulsive effects, stochastic effects likewise exist in real systems. It is well known that a lot of dynamic systems have variable structures subject to stochastic abrupt changes, which may result from abrupt phenomena such as stochastic failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, etc. (see [3,2,4–6] and references therein). Therefore, the investigation of impulsive stochastic differential equations attracts great attention, especially as regards stability (see [7,8] and references therein).

Motivated by the results in Yang and Xu [2] concerning mean square exponential stability of impulsive stochastic difference equations with discrete time, we will, in this present work, be interested in exponential stability in mean square of stochastic difference equations with continuous time.

[☆] This work was partially supported by NNSF of China (Grant No. 10671212).

* Corresponding author.

E-mail address: jianhaibao@yahoo.com.cn (J. Bao).

2. Impulsive stochastic difference equations with continuous time

Let $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P}\}$ be a probability space with a filtration satisfying the usual conditions, i.e., the filtration is continuous on the right and \mathcal{F}_0 contains all \mathbf{P} -zero sets. Let $\|\varphi\| = \sup_{s \in \Theta} |\varphi(s)|$ hold.

In the present work, we consider a class of impulsive stochastic difference equations with continuous time as follows:

$$\begin{cases} x(t + \tau) = F(t, x(t), x(t - h_1), \dots, x(t - h_m)) + G(t, x(t), x(t - h_1), \dots, x(t - h_m))\xi(t), & t \neq \tau_k, \\ x(\tau_k) = H(\tau_k, x(\tau_k^-)), & k = 1, 2, \dots, \quad t = \tau_k, \end{cases} \quad (2.1)$$

with initial condition

$$x(\theta) = \phi(\theta), \quad \theta \in \Theta = [t_0 - \tau - h_m, t_0].$$

The fixed moments of time τ_k satisfy $0 \leq \tau_1 < \tau_2 < \dots < \tau_k < \dots$, $\lim_{k \rightarrow \infty} \tau_k = \infty$. $\xi(t) \in R$ is a $\{\mathcal{F}_t\}$ -measurable stationary and mutually independent stochastic process satisfying

$$E\xi(t) = 0, \quad E\xi^2(t) = 1.$$

Moreover, $\xi(t)$ is independent on F and G .

Assume that

$$F : [t_0 - \tau, \infty) \times R^{m+1} \rightarrow R, \quad G : [t_0 - \tau, \infty) \times R^{m+1} \rightarrow R, \quad H : [0, \infty) \times R \rightarrow R.$$

In this work, we always assume that under certain conditions the system (2.1) admits a unique solution which is continuous on the right and limitable on the left. Furthermore, assume that $F(t, 0, \dots, 0) = 0$, $G(t, 0, \dots, 0) = 0$ and $H(\tau_k, 0) = 0$, $k = 1, 2, \dots$. Hence, Eq. (2.1) has a null solution.

Now, we introduce the definition of exponential stability in mean square.

Definition 2.1. The null solution of Eq. (2.1) is called exponential stability in mean square if there exist two positive constants λ and M such that

$$E|x(t)|^2 \leq M\|\varphi\|^2 e^{-\lambda t}, \quad t \geq t_0 - \tau.$$

Furthermore, we impose the following assumptions.

(H1) For any $t \geq t_0 - \tau$, there exist two nonnegative functions $a_j(t)$ and $b_j(t)$ such that

$$|F(t, x(t), x(t - h_1), \dots, x(t - h_m))| \leq \sum_{j=0}^m a_j(t)|x(t - h_j)|$$

and

$$|G(t, x(t), x(t - h_1), \dots, x(t - h_m))| \leq \sum_{j=0}^m b_j(t)|x(t - h_j)|,$$

where $h_0 = 0$.

(H2) $\sup_{t \geq 0} \{a^2(t) + b^2(t)\} = \mu < 1$, with $a(t) = \sum_{j=0}^m a_j(t)$ and $b(t) = \sum_{j=0}^m b_j(t)$.

(H3) There exist constants $d_k \geq 1$ which satisfy

$$|H(\tau_k, x(\tau_k^-))| \leq d_k|x(\tau_k^-)|, \quad k = 1, 2, \dots$$

(H4) There is a constant $\alpha \geq 0$ such that

$$\frac{2 \ln d_k}{\tau_k - \tau_{k-1}} \leq \alpha < \lambda, \quad k = 1, 2, \dots,$$

where $\tau_0 = 0$ and λ satisfies

$$0 < \lambda \leq \frac{1}{\tau + h_m} \ln \frac{1}{\mu}.$$

3. Main result

In this section, we shall present the main result and complete the proof. Unlike earlier studies, ours does not make use of general methods such as Lyapunov methods, Itô formula methods and so forth. However, we firstly establish a difference inequality with continuous time, which plays an important role in the following section, for obtaining our desired result. Because of the difficulties brought in by impulsive and stochastic effects, we need to estimate the solution of Eq. (2.1) on $[\tau_k, \tau_{k+1})$ with initial conditions on $[\tau_k - h_m - \tau, \tau_k]$.

For convenience, we introduce the following lemma.

Lemma 3.1. *Let $c_j(t) \in R^+, j = 1, 2, \dots$, and $\sup_{t \geq 0} \{\sum_{j=0}^m c_j(t)\} = \eta < 1$ hold. Moreover, assume that the function $u(t)$ satisfies the difference inequality with continuous time*

$$u(t + \tau) \leq \sum_{j=0}^m c_j(t)u(t - h_j), \quad t \geq t_0 \geq 0. \quad (3.1)$$

Thus

$$u(t) \leq de^{-\lambda t}, \quad t \geq t_0 \geq 0, \quad (3.2)$$

provided the initial condition satisfies the inequality

$$u(t) \leq de^{-\lambda t}, \quad t \in \Theta = [t_0 - \tau - h_m, t_0], \quad (3.3)$$

where $t_0 \in R^+$ and λ satisfies

$$0 < \lambda \leq \frac{1}{\tau + h_m} \ln \frac{1}{\eta}. \quad (3.4)$$

Proof. We define

$$y(t) = u(t)e^{\lambda t}. \quad (3.5)$$

This, together with (3.3), (3.5), yields

$$y(t) \leq d, \quad t \in [t_0 - \tau - h_m, t_0].$$

Presently, for any $t \geq t_0$, we will show that

$$y(t) \leq d. \quad (3.6)$$

If inequality (3.6) is not true, then there exists a $t^* + \tau > t_0$ such that

$$y(t^* + \tau) > d, \quad \text{however,} \quad y(t) \leq d, \quad t \in [t_0 - \tau - h_m, t^* + \tau). \quad (3.7)$$

In view of (3.1), (3.4) and (3.7), we derive that

$$\begin{aligned} y(t^* + \tau) &= u(t^* + \tau)e^{\lambda(t^* + \tau)} \\ &\leq e^{\lambda(t^* + \tau)} \sum_{j=0}^m c_j(t^*)u(t^* - h_j) \\ &= e^{\lambda(t^* + \tau)} \sum_{j=0}^m c_j(t^*)y(t^* - h_j)e^{-\lambda(t^* - h_j)} \\ &\leq e^{\lambda(h_m + \tau)} \sum_{j=0}^m c_j(t^*)y(t^* - h_j) \\ &\leq de^{\lambda(h_m + \tau)} \sum_{j=0}^m c_j(t^*) \\ &\leq de^{\lambda(h_m + \tau)} \eta \\ &\leq d, \end{aligned} \quad (3.8)$$

which contradicts the first inequality of (3.7). Therefore, for any $t \geq t_0$,

$$y(t) \leq d.$$

The desired result is obtained. \square

Theorem 3.1. Under the assumptions (H_1) – (H_4) , then the null solution of Eq. (2.1) is exponential stability in mean square and the exponential convergence rate is equal to $\lambda - \alpha$.

Proof. From assumption (H_1) and the Hölder inequality, we obtain that

$$\begin{aligned} Ex^2(t + \tau) &= EF^2(t, x(t), x(t - h_1), \dots, x(t - h_m)) + EG^2(t, x(t), x(t - h_1), \dots, x(t - h_m)) \\ &\leq E \left(\sum_{j=0}^m a_j(t) |x(t - h_j)| \right)^2 + E \left(\sum_{j=0}^m b_j(t) |x(t - h_j)| \right)^2 \\ &\leq \sum_{j=0}^m a_j(t) \sum_{j=0}^m a_j(t) E |x(t - h_j)|^2 + \sum_{j=0}^m b_j(t) \sum_{j=0}^m b_j(t) E |x(t - h_j)|^2 \\ &= \sum_{j=0}^m [a(t)a_j(t) + b(t)b_j(t)] E |x(t - h_j)|^2, \quad t \neq \tau_k, k = 1, 2, \dots \end{aligned} \quad (3.9)$$

With the assumption (H_2) , this yields

$$\sup_{t \geq 0} \left\{ \sum_{j=0}^m [a(t)a_j(t) + b(t)b_j(t)] \right\} = \sup_{t \geq 0} \{a^2(t) + a^2(t)\} = \mu < 1.$$

By virtue of the initial condition $x(\theta) = \phi(\theta)$, $\theta \in \Theta = [-\tau - h_m, 0]$, we get

$$Ex^2(t) \leq \|\phi\|^2 e^{-\lambda t}, \quad t \in \Theta = [-\tau - h_m, 0].$$

Therefore, by Lemma 3.1,

$$Ex^2(t) \leq \|\phi\|^2 e^{-\lambda t}, \quad t \in [0, \tau_1).$$

In the following, we will prove that for all $k = 2, 3, \dots$,

$$Ex^2(t) \leq d_0^2 d_1^2 \dots d_{k-1}^2 \|\phi\|^2 e^{-\lambda t}, \quad t \in [\tau_{k-1}, \tau_k), \quad (3.10)$$

with $d_0 = 1$ and $i_0 = 0$. Suppose, for all $k = 1, 2, \dots, l$,

$$Ex^2(t) \leq d_0^2 d_1^2 \dots d_{k-1}^2 \|\phi\|^2 e^{-\lambda t}, \quad t \in [\tau_{k-1}, \tau_k), \quad (3.11)$$

with $d_0 = 1$ and $\tau_0 = 0$. By assumption (H_3) , together with (3.11), we have

$$\begin{aligned} Ex^2(\tau_l) &= E |H(\tau_l, x(\tau_l^-))|^2 \\ &\leq d_l^2 E |x(\tau_l^-)|^2 \\ &\leq d_0^2 d_1^2 \dots d_{l-1}^2 d_l^2 \|\phi\|^2 e^{-\lambda \tau_l}. \end{aligned}$$

Hence,

$$Ex^2(t) \leq d_0^2 d_1^2 \dots d_{l-1}^2 d_l^2 \|\phi\|^2 e^{-\lambda t}, \quad t \in [\tau_l - h_m - \tau, \tau_l],$$

which implies, by Lemma 3.1,

$$Ex^2(t) \leq d_0^2 d_1^2 \dots d_{l-1}^2 d_l^2 \|\phi\|^2 e^{-\lambda t}, \quad t \in [\tau_l, \tau_{l+1}).$$

By mathematical induction, (3.10) is true for all $k = 2, 3, \dots$. In view of (H_4) , this yields $d_k^2 \leq e^{\alpha(\tau_k - \tau_{k-1})}$ with $k = 1, 2, \dots$; thus, for any $t \in [\tau_{k-1}, \tau_k)$,

$$\begin{aligned} Ex^2(t) &\leq e^{\alpha(\tau_1 - \tau_0)} e^{\alpha(\tau_2 - \tau_1)} \dots e^{\alpha(\tau_{k-1} - \tau_{k-2})} \|\phi\|^2 e^{-\lambda t} \\ &= \|\phi\|^2 e^{-\lambda t + \alpha \tau_{k-1}} \\ &\leq \|\phi\|^2 e^{-(\lambda - \alpha)t}. \end{aligned}$$

The desired result is obtained. \square

Remark 3.1. In Eq. (2.1), provided $H(\tau_k, x(\tau_k^-)) = x(\tau_k^-)$, then the impulsive stochastic difference equation with continuous time becomes a stochastic difference equation with continuous time and without impulsive effects, that is to say, our Theorem 3.1 is effective for it.

4. An illustrative example

To illustrate the effectiveness of our theorem, we establish an example.

Example 4.1. We investigate a class of impulsive stochastic difference equations with continuous time as follows:

$$\begin{cases} x(t+1) = \frac{1}{4} \sin(x(t)) - \frac{1}{3}x(t-1) + \frac{1}{2}x(t)\xi(t), & t \neq \tau_k, \\ x(\tau_k) = e^{0.05\tau_k}x(\tau_k^-), & t = \tau_k. \end{cases} \quad (4.1)$$

Proof. We define

$$\begin{aligned} F(t, x(t), x(t-h_1), \dots, x(t-h_m)) &= \frac{1}{4} \sin(x(t)) - \frac{1}{3}x(t-1) \\ G(t, x(t), x(t-h_1), \dots, x(t-h_m)) &= \frac{1}{2}x(t), \quad H(\tau_k, x(\tau_k^-)) = e^{0.05\tau_k}x(\tau_k^-). \end{aligned}$$

Obviously,

$$\begin{aligned} |F(t, x(t), x(t-h_1), \dots, x(t-h_m))| &\leq \frac{1}{4}|x(t)| + \frac{1}{3}|x(t-1)| \\ |G(t, x(t), x(t-h_1), \dots, x(t-h_m))| &\leq \frac{1}{2}|x(t)|, \quad |H(\tau_k, x(\tau_k^-))| = e^{0.05\tau_k}|x(\tau_k^-)|. \end{aligned}$$

Therefore, the parameters of assumptions (H₁)–(H₃) are as follows:

$$\begin{aligned} a_0(t) &= \frac{1}{4}, \quad a_1(t) = \frac{1}{3}, \quad b_0(t) = \frac{1}{2}, \quad b_1(t) = 0, \quad t > -1, \\ \mu &= \frac{87}{144} < 1, \quad d_k = e^{0.05\tau_k} > 1, \quad k = 1, 2, \dots \end{aligned}$$

Let $\alpha = \frac{2 \ln d_k}{\tau_k - \tau_{k-1}} = 0.1$ and $\lambda = 0.26 \leq \frac{1}{2} \ln \frac{1}{\mu} = \frac{1}{2} \ln \frac{144}{87} = 0.265314125$. Hence the assumption (H₄) is satisfied. By Theorem 3.1, we derive that the null solution of Eq. (4.1) is exponential stability in mean square and the exponential convergence rate is equal to 0.16. \square

Acknowledgements

The authors would like to thank the referee and the associated editor for their useful comments and suggestions.

References

- [1] V. Lakshmikantham, D. Bainov, P. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.
- [2] Zhiguo Yang, Daoyi Xu, Mean square exponential stability of impulsive difference equations, *Applied Mathematics Letters* 20 (2007) 938–945.
- [3] Zhiguo Yang, Daoyi Xu, Li Xiang, Exponential p -stability of impulsive stochastic differential equations with delay, *Physical Letters A* 359 (2006) 129–137.
- [4] Shengyuan Xu, Tongwen Chen, Robust filtering for uncertain impulsive stochastic systems under sampled measurements, *Automatica* 39 (2003) 509–516.
- [5] Huijun Wu, Jitao Sun, p -moment stability of stochastic differential equations with impulsive jump and Markovian switching, *Automatica* 42 (2006) 1753–1759.
- [6] Shijin Wu, Dong Han, Xianzhang Meng, p -moment stability of stochastic differential equations with jumps, *Applied Mathematics and Computation* 152 (2004) 505–519.
- [7] X. Mao, Stability of stochastic differential equations with Markovian switching, *Stochastic Processes and Application* 79 (1) (1999) 45–67.
- [8] O.M. Kwon, J.H. Par, Exponential stability for uncertain cellular neural networks with discrete and distributed time-varying delays, *Applied Mathematics and Computation* 203 (2) (2008) 813–823.
- [9] G. Ladas, Recent development in the oscillation of delay difference equations, in: *Differential Equations*, (Colorado Springs, CO, 1989), in: *Lecture Notes in Pure and Appl. Math.*, vol. 127, Dekker, New York, 1991, pp. 321–332.
- [10] Yu.L. Maistrenko, A.N. Sharkovsky, Difference equations with continuous time as mathematical models of the structure emergences, in: *Dynamical Systems and Environmental Models*, in: Eisenach, Mathem. Ecol., Akademie-Verlag, Berlin, 1986, pp. 40–49.
- [11] M.G. Blizorukov, On the construction of solutions of linear difference systems with continuous time, *Differentsialniye Uravneniya* 32 (1996) 127–128. Translation in *Diff. Eqs.* 32 (1996) 133–134.
- [12] Y. Domstlak, Oscillatory properties of linear difference equations with continuous time, *Differential Equations Dynamical Systems* 1 (4) (1993) 311–324.
- [13] L. Shaikhet, Necessary and sufficient conditions of asymptotic mean square stability for linear difference equations, *Applied Mathematics Letters* 10 (3) (1997) 111–115.
- [14] L. Shaikhet, Lyapunov functionals construction for stochastic difference second kind Volterra equations with continuous time, *Advance in Difference Equations* 2004 (1) (2004) 67–91.